

**EFFECTIVE CONSTRAINT POTENTIAL FOR
ABELIAN MONOPOLE IN $SU(2)$ LATTICE GAUGE THEORY**M.N. CHERNODUB¹, M.I. POLIKARPOV^{1,2} and A.I. VESELOV¹¹*ITEP, B.Chermushkinskaya 25, Moscow, 117259, Russia*²*Department of Physics, Kanazawa University, Kanazawa 920-11, Japan***Abstract**

We describe numerical calculation results for the probability distribution of the value of the monopole creation operator in the $SU(2)$ lattice gluodynamics. We work in the maximal abelian projection. It occurs that at low temperatures, below the deconfinement phase transition, the maximum of the distribution is shifted from zero, which means that the effective constraint potential is of the Higgs type. Above the phase transition the minimum of the potential (the maximum of the monopole field distribution) is at the zero value of the monopole field. This fact confirms the existence of the abelian monopole condensate in the confinement phase of lattice gluodynamics, and agrees with the dual superconductor model of the confining vacuum.

1 Introduction

The monopole mechanism of the color confinement [1, 2, 3] is generally accepted by the lattice community. Still there are many open questions [4]. In the lattice gluodynamics it is very important to find the order parameter, constructed from the monopole field for the deconfinement phase transition. The first candidate is the value of the monopole condensate, which should be nonzero in the confinement phase and vanish at the phase transition. To study the monopole condensate, we need an explicit expression for the operator $\Phi_{mon}(x)$, which creates the abelian monopole at the point x . The operator $\Phi_{mon}(x)$, found for the compact electrodynamics with the Villain form of the action by Fröhlich and Marchetti [5], was studied numerically in [6]. In Section 2 we construct the analogous monopole creation operator for an arbitrary abelian projection of the lattice $SU(2)$ gluodynamics. The numerical results presented in Section 3 are obtained for the maximal abelian projection. As shown by many numerical simulations for this projection the gluodynamic vacuum behaves as the dual superconductor (see reviews [7, 8] and references therein). In [9] another form of the monopole creation operator was studied; it is shown that its expectation value vanishes in the deconfinement phase. The monopole creation operator, constructed in Section 2 is positive, and therefore its expectation value cannot vanish at $T = T_c$. Still the results of the numerical studies of the effective potential for Φ_{mon} presented in Section 3, clearly indicate that the monopole condensate exists in the

confinement phase of lattice gluodynamics and does not exist in the deconfinement phase. The analogous claim is done in ref.[10], where the monopole condensate is calculated on the basis of the percolation properties of the monopole currents. The monopole creation operator in the monopole current representation is studied in ref.[11]. Again, the monopole creation operator depends on the temperature as the disorder parameter.

In Appendices A and B we prove that the operator used in the numerical calculations create the monopole in the lattice $U(1)$ theory with the general form of the action. Appendix C contains the brief description of the differential form notations on the lattice, these notations are used in Appendices A and B.

2 Monopole Creation Operator

First we give a formal construction of the monopole creation operator for the abelian projection of the $SU(2)$ gluodynamics. Let us parametrize the $SU(2)$ link matrix in the standard way: $U_{x\mu}^{11} = \cos \phi_{x\mu} e^{i\theta_{x\mu}}$; $U_{x\mu}^{12} = \sin \phi_{x\mu} e^{i\chi_{x\mu}}$; $U_{x\mu}^{22} = U_{x\mu}^{11*}$; $U_{x\mu}^{21} = -U_{x\mu}^{12*}$; $0 \leq \phi \leq \pi/2$, $-\pi < \theta, \chi \leq \pi$. The plaquette action in terms of the angles ϕ , θ and χ can be written as follows:

$$S_P = \frac{1}{2} \text{Tr} U_1 U_2 U_3^+ U_4^+ = S^a + S^n + S^i, \quad (1)$$

where

$$S^a = \cos \theta_P \cos \phi_1 \cos \phi_2 \cos \phi_3 \cos \phi_4, \quad (2)$$

S^n and S^i describe the interaction of the fields θ and χ and selfinteraction of the field χ [12]; $\theta_P = \theta_1 + \theta_2 - \theta_3 - \theta_4$, here the subscripts 1, ..., 4 correspond to the links of the plaquette: $1 \rightarrow \{x, x + \hat{\mu}\}$, ..., $4 \rightarrow \{x, x + \hat{\nu}\}$. For a fixed abelian projection, each term S^a , S^n and S^i is invariant under the residual $U(1)$ gauge transformations:

$$\theta_{x\mu} \rightarrow \theta_{x\mu} + \alpha_x - \alpha_{x+\hat{\mu}}, \quad (3)$$

$$\chi_{x\mu} \rightarrow \chi_{x\mu} + \alpha_x + \alpha_{x+\hat{\mu}}. \quad (4)$$

We define the operator which creates the monopoles at the point x of the dual lattice as follows:

$$\Phi_{mon}(x) = \exp \{ \beta [-S(\theta_P, \dots) + S(\theta_P + W_P(x), \dots)] \}, \quad (5)$$

the function $W_P(x)$ being defined by eq. (A.6). Substituting (1–2) into (5), we get

$$\Phi_{mon}(x) = \exp \left\{ \sum_P \tilde{\beta} [-\cos(\theta_P) + \cos(\theta_P + W_P(x))] \right\}, \quad (6)$$

where $\tilde{\beta} = \cos \phi_1 \cos \phi_2 \cos \phi_3 \cos \phi_4 \beta$. Effectively the monopole creation operator shifts all abelian plaquette angles θ_P .

For the compact electrodynamics with the Villain type of the action the above definition coincides with the definition of Fröhlich and Marchetti [5]. For the general type of the action in compact electrodynamics we can use the described above construction. The proof is outlined in Appendices A,B. The gluodynamics in the abelian projection contains the compact gauge field θ and the charged *vector* field χ . The action (1) in terms of the fields θ and χ is rather nontrivial, and at the moment we can not prove that the above construction of the monopole creation operator is valid in this case. However, there is a proof for a similar Abelian – Higgs model, with the general type of the action, and this proof is analogous to one given in Appendix B. Moreover the numerical results, presented in the next section, clearly show that the introduced operator is the order parameter for the deconfinement phase transition.

3 Numerical Results

The numerical results are obtained on the lattice $4 \cdot L^3$, for $L = 8, 10, 12, 14, 16$. The preliminary numerical results are published in ref. [13]. We extrapolate our results to the infinite volume since near the phase transition the finite volume effects are very strong. We impose anti-periodic boundary conditions in space directions for the abelian fields, since the construction of the operator Φ_{mon} can be done only in the time slice with the anti-periodic boundary conditions. Periodic boundary conditions are forbidden due to the Gauss law: we input a magnetic charge into the finite box. Formally, equation (A.5) for $*D$ admits no solution in the finite box with periodic boundary conditions. To impose anti-periodic boundary conditions on the abelian fields the C -periodic boundary conditions should be imposed on the nonabelian gauge fields [14]. In the case of $SU(2)$ gauge group the C -periodic boundary conditions are almost trivial: on the boundary we have $U_{x,\mu} \rightarrow \Omega^+ U_{x,\mu} \Omega$, $\Omega = i\sigma_2$. To get the order parameter for the deconfinement phase transition we study the probability distribution of the operator Φ_{mon} ; we calculate the expectation value $\langle \delta(\Phi - \Phi_{mon}(x)) \rangle$. The effective constraint potential,

$$V_{eff}(\Phi) = -\ln(\langle \delta(\Phi - \frac{1}{V} \sum_x \Phi_{mon}(x)) \rangle) \quad (7)$$

has more physical meaning than the probability distribution. The calculation of $V_{eff}(\Phi)$ is time consuming, and we present our results for $V(\Phi)$, defined as follows:

$$V(\Phi) = -\ln(\langle \delta(\Phi - \Phi_{mon}(x)) \rangle). \quad (8)$$

In Figs. 1(a) and 1(b) $V(\Phi)$ is shown for the confinement and the deconfinement phases, the calculations being performed on the lattice $4 \cdot 12^3$.

In the confinement phase the minimum of $V(\Phi)$ is shifted from zero, while in the deconfinement phase the minimum is at the zero value of the monopole field Φ . We have used the positive operator, $\Phi_{mon}(x) > 0$ (6), however in the dual representation the creation operator of the monopole (A.5) is not positively definite: the sign is lost when we perform the inverse duality transformation (cf. eq. (A.5) with eq. (A.6)). The potential shown in Fig. 1(a) corresponds to the Higgs type potential. The value of the monopole

field, Φ_c , at which the potential has a minimum is equal to the value of the monopole condensate. The potential shown in Fig. 1(b) corresponds to the trivial potential with a minimum at the zero value of the field: $\Phi_c = 0$. The dependence of the minimum of the potential, Φ_c , on the spatial size of the lattice, L , is shown in Fig. 2. The gauge fields are generated by the standard heat bath method. At each value of β 2000 update sweeps are performed to thermalize the system. The maximal abelian projection [3] corresponds to the maximization of the quantity $R = \sum_{x,\mu} \text{Tr}(U_{x\mu}\sigma_3 U_{x\mu}^+ \sigma_3)$. For our relatively small lattices the overrelaxation algorithm [17, 18] and simple local maximization method [3] give approximately the same results, and we use the local maximization method. We stop our gauge fixing sweeps when $Z = 10^{-5}$, here Z [18] is the lattice analogue of the quantity $< |(\partial_\mu + iA_\mu^3)A_\mu^+|^2 >$ which should be zero if the maximal abelian projection is fixed exactly. We checked that the more precise gauge fixing do not change the results for Φ_c inside the statistical errors. For each value of β at the lattice of definite size we use 100 gauge field configurations separated by 300 Monte Carlo sweeps. For each configuration we calculated the value of the monopole creation operator at 20 randomly chosen lattice points. Therefore for each value of β at the lattice of definite size we have 2000 values of Φ_{mon} . We use these values to calculate the quantity Φ_c (the maximum of the probability distribution $\rho(\Phi_{mon})$ is at $\Phi_{mon} = \Phi_c$).

We fitted the data for Φ_c by the formula $\Phi_c = AL^\alpha + \Phi_c^{inf}$, where A , α and Φ_c^{inf} are the fitting parameters. It occurs that $\alpha = -1$ within statistical errors. Fig. 3 shows the dependence on β of the value of the monopole condensate, extrapolated to the infinite spatial volume, Φ_c^{inf} . It is clearly seen that Φ_c^{inf} vanishes at the point of the phase transition and it plays the role of the order parameter.

Acknowledgments

The authors are grateful to P. van Baal, A. Di Giacomo, R. Haymaker, T. Ivanenko, Y. Matsubara, A. Pochinsky, T. Suzuki and U. -J. Wiese for useful discussions. M.I.P. feels much obliged for the kind reception given to him by the staff of Theoretical Physics Institute, the University of Minneapolis, and the staff of the Department of Physics of the Kanazawa University, where the part of this work has been done. This work was supported by the JSPS Program on Japan – FSU scientists collaboration, by the Grant INTAS-94-0840, and by the grant number 96-02-17230a, financed by the Russian Foundation for Fundamental Sciences.

Appendix A

Here we construct the monopole creation operator for the compact electrodynamics with the general type of the action. A similar construction exists for the compact Abelian–Higgs model with the general type of the action. First we perform the duality transformation of the partition function for the $4D$ lattice compact electrodynamics with the general type of the action $\mathcal{S}(d\theta + 2\pi) = \mathcal{S}(d\theta)$,

$$\mathcal{Z} = \int_{-\pi}^{+\pi} \mathcal{D}\theta \exp \{-\mathcal{S}(\mathrm{d}\theta)\}, \quad (\text{A.1})$$

We use the notations of the calculus of differential forms on the lattice [19] (see also Appendix C). The symbol $\int \mathcal{D}\theta$ denotes the integral over all link variables θ . First consider the Fourier series for the Boltzmann factor:

$$\mathcal{Z} = \text{const.} \int_{-\pi}^{+\pi} \mathcal{D}\theta \sum_{n(\mathbf{c}_2) \in \mathbb{Z}} F(n) e^{i(n, \mathrm{d}\theta)}, \quad F(n) = \int_{-\pi}^{+\pi} \mathcal{D}X \exp \{-\mathcal{S}[X] - i(n, X)\}. \quad (\text{A.2})$$

Integrating over θ we get the partition function of the dual theory:

$$\mathcal{Z}^d = \sum_{k(\mathbf{c}_2) \in \mathbb{Z}} \exp \{-^* \mathcal{S}(\mathrm{d}^* k)\}, \quad (\text{A.3})$$

where $n = \delta k$, $^* \mathcal{S}(n) = -\ln F(n)$. We can represent \mathcal{Z} as the following limit of the partition function for the Abelian–Higgs theory:

$$\mathcal{Z}^d = \lim_{\eta \rightarrow \infty} \int_{-\pi}^{+\pi} \mathcal{D}^* \varphi \int_{-\infty}^{+\infty} \mathcal{D}^* B \sum_{^* k(\mathbf{c}_1) \in \mathbb{Z}} \exp \{-^* \mathcal{S}(\mathrm{d}^* B/2\pi) - \eta \| ^* B - \mathrm{d}^* \varphi + 2\pi^* k \|^2\}, \quad (\text{A.4})$$

here $^* \mathcal{S}(\mathrm{d}^* B/2\pi)$ is the kinetic energy of the dual gauge field $^* B$ (the analogue of $\tilde{F}_{\mu\nu}^2$) and the Higgs field $\exp\{i^* \varphi\}$ carries magnetic charge, since it interacts via the covariant derivative with the dual gauge field $^* B$. The Dirac operator [15],

$$\Phi_{mon}^d(x) = e^{i^* \varphi} \cdot \exp \{-i(^* D_x, ^* B)\}, \quad \delta^* D_x = ^* \delta_x \quad (\text{A.5})$$

is the gauge invariant monopole creation operator. It creates the cloud of photons and the monopole at the point x . In (A.5) $^* \delta_x$ is the lattice δ -function, it equals to unity at the site x of the dual lattice and is zero at the other sites. Note that in the above formulas the radial part of the Higgs field which carries the magnetic charge is fixed to unity.

Coming back to the original partition function (A.1) we get the expectation value of the monopole creation operator in terms of the fields θ :

$$\langle \Phi_{mon}(x) \rangle = \frac{1}{\mathcal{Z}} \int_{-\pi}^{+\pi} \mathcal{D}\theta \exp \{-\mathcal{S}(\mathrm{d}\theta + W_P)\}, \quad W_P = 2\pi \delta \Delta^{-1}(D_x - \omega_x), \quad (\text{A.6})$$

where the Dirac string attached to the monopole [5], is represented by the integer valued 1-form $^* \omega_x$, which satisfies the equation: $\delta^* \omega_x = ^* \delta_x$.

Appendix B

Here we represent the partition function (A.1) for the compact electrodynamics with the general type of the action as the sum over the monopole currents. First we insert the unity $1 = \int_{-\infty}^{+\infty} \mathcal{D}G \delta(G - n)$ into the sum (A.2) and use the Poisson summation formula: $\sum_n \delta(G - n) = \sum_n e^{2\pi i(G, n)}$. We get:

$$\mathcal{Z} = \text{const} \cdot \int_{-\pi}^{+\pi} \mathcal{D}\theta \int_{-\infty}^{+\infty} \mathcal{D}G \sum_{n(c_2) \in \mathbb{Z}} F(G) \exp \{i(d\theta + 2\pi n, G)\} . \quad (\text{B.1})$$

Here G is a real-valued two-form. It is possible to change the summation variable n : $\sum_n f(n) = \sum_q \sum_{\delta^* j=0} f(m[j] + dq)$, where $n = m[j] + dq$, $dm[j] = j$, $dj = 0$. Now we

change the compact integration variable, θ , to the noncompact one A : $\sum_n \int_{-\pi}^{+\pi} \mathcal{D}\theta f(d\theta + 2\pi n) = \sum_{\delta^* j=0} \int_{-\infty}^{+\infty} \mathcal{D}A f(dA + 2\pi \delta \Delta^{-1} j)$, where $A = \theta + 2\pi \Delta^{-1} \delta m[j] + 2\pi q$ and we use the

Hodge-de-Rahm formula: $m = \delta \Delta^{-1} j + d\Delta^{-1} \delta m$. The integral over A gives the constraint $\delta(\delta G)$, which we solve introducing the new integration variable H , $G = \delta H$. Taking into account the relation $d\delta \Delta^{-1} j = j$, valid for any j , such that $dj = 0$, we finally get the representation of the partition function as a sum over the conserved monopole currents:

$$\mathcal{Z} = \text{const} \cdot \sum_{\substack{*j(*c_1) \in \mathbb{Z} \\ \delta^* j=0}} e^{-\mathcal{S}_{mon}(*j)} . \quad (\text{B.2})$$

where

$$\mathcal{S}_{mon}(*j) = -\ln \left(\int_{-\infty}^{+\infty} \mathcal{D}H F(\delta H) \exp \{2\pi i(*H, *j)\} \right) . \quad (\text{B.3})$$

The monopole action is nonlocal due to the integral over H . If we start from the Villain action $\mathcal{S}^V(d\theta) = -\ln \sum_n \exp \{-\beta \|d\theta + 2\pi n\|^2\}$, then the integral over H in (B.3) is Gaussian, and we get the well known expression [16] for the monopole action: $\mathcal{S}_{mon}^V(*j) = 4\pi^2 \beta (*j, \Delta^{-1} *j)$. Using transformations similar to (B.1)–(B.2) for the expectation value of the monopole creation operator (5), we get

$$\langle \Phi_{mon}(x) \rangle = \frac{1}{\mathcal{Z}} \sum_{\delta^* j = \delta_x} \exp \{\mathcal{S}_{mon}(*j - *D_x)\} , \quad (\text{B.4})$$

where D_x is defined by eq. (A.5). Therefore $\Phi_{mon}(x)$ creates a non-closed monopole world trajectory starting at the point x , which shows that, indeed, $\Phi_{mon}(x)$ is a monopole creation operator.

Appendix C

Let us briefly summarize the main notions from the theory of differential forms on the lattice [19]. The advantage of the calculus of differential forms consists in the general character of the expressions obtained. Most of the transformations depend neither on the space-time dimension, nor on the rank of the fields. With minor modifications, the transformations are valid for lattices of any form (triangular, hypercubic, random, *etc*). A differential form of rank k on the lattice is a function ϕ_k defined on k -dimensional cells c_k of the lattice, *e.g.*, the scalar (gauge) field is a 0-form (1-form). The exterior differential operator d is defined as follows:

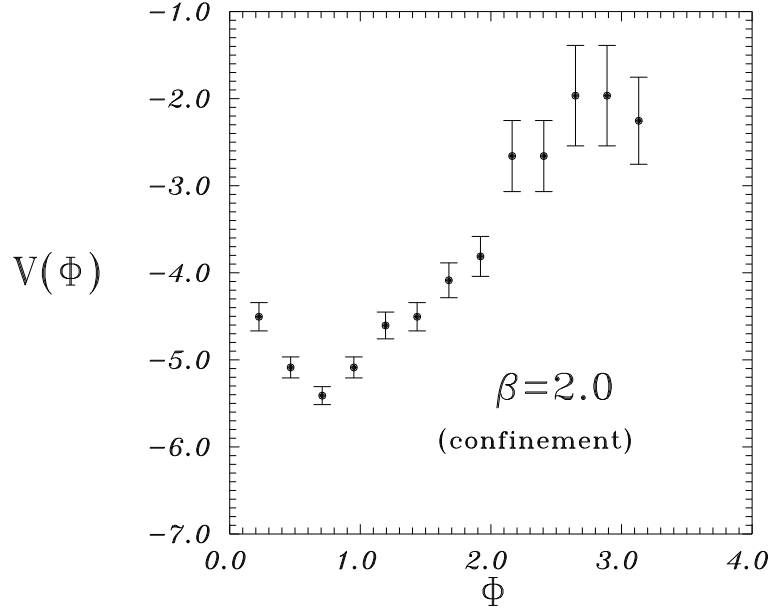
$$(d\phi)(c_{k+1}) = \sum_{c_k \in \partial c_{k+1}} \phi(c_k). \quad (\text{C.1})$$

Here ∂c_k is the oriented boundary of the k -cell c_k . Thus the operator d increases the rank of the form by unity; $d\varphi$ is the link variable constructed, as usual, in terms of the site angles φ , and dA is the plaquette variable constructed from the link variables A . The scalar product is defined in the standard way: if φ and ψ are k -forms, then $(\varphi, \psi) = \sum_{c_k} \varphi(c_k) \psi(c_k)$, where \sum_{c_k} is the sum over all cells c_k . To any k -form on the D -dimensional lattice there corresponds a $(D-k)$ -form ${}^*\Phi({}^*c_k)$ on the dual lattice, *c_k being the $(D-k)$ -dimensional cell on the dual lattice. The co-differential $\delta = {}^*d^*$ satisfies the partial integration rule: $(\varphi, \delta\psi) = (d\varphi, \psi)$. Note that $\delta\Phi(c_k)$ is a $(k-1)$ -form and $\delta\Phi(c_0) = 0$. The norm is defined by: $\|a\|^2 = (a, a)$; therefore, $\|B - d\varphi + 2\pi n\|^2$ in (A.4) implies summation over all links. $\sum_{l(c_1) \in \mathbb{Z}}$ denotes the sum over all configurations of the integers l attached to the links c_1 . The action (A.4) is invariant under the gauge transformations $B' = B + d\alpha$, $\varphi' = \varphi + \alpha$ due to the well known property $d^2 = \delta^2 = 0$. The lattice Laplacian is defined by: $\Delta = d\delta + \delta d$.

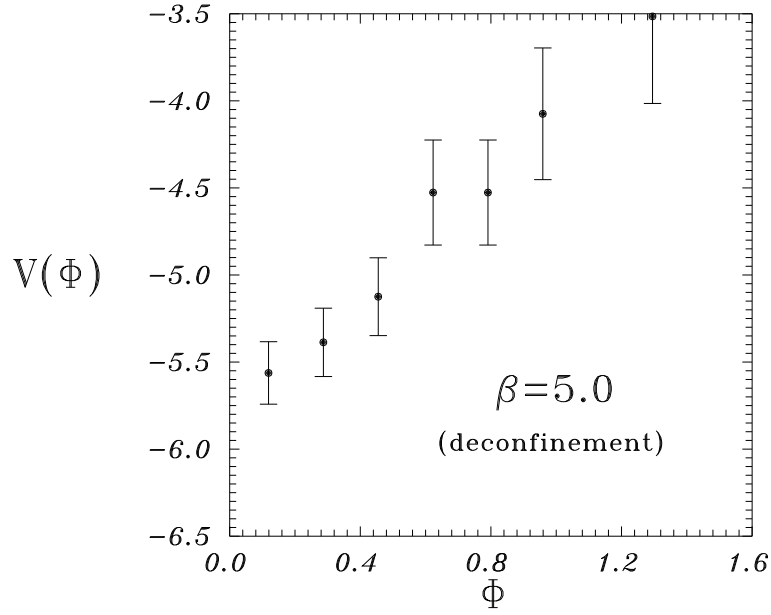
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(a)



(b)

Figure 1: $V(\Phi)$ for the confinement (a) and the deconfinement (b) phases.

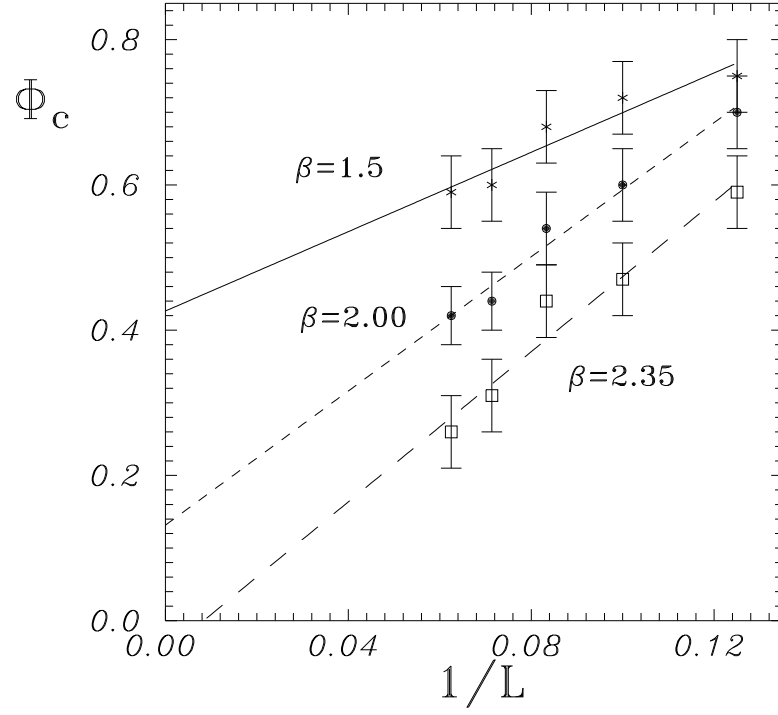


Figure 2: The dependence of Φ_c on the spatial size of the lattice for three values of β .

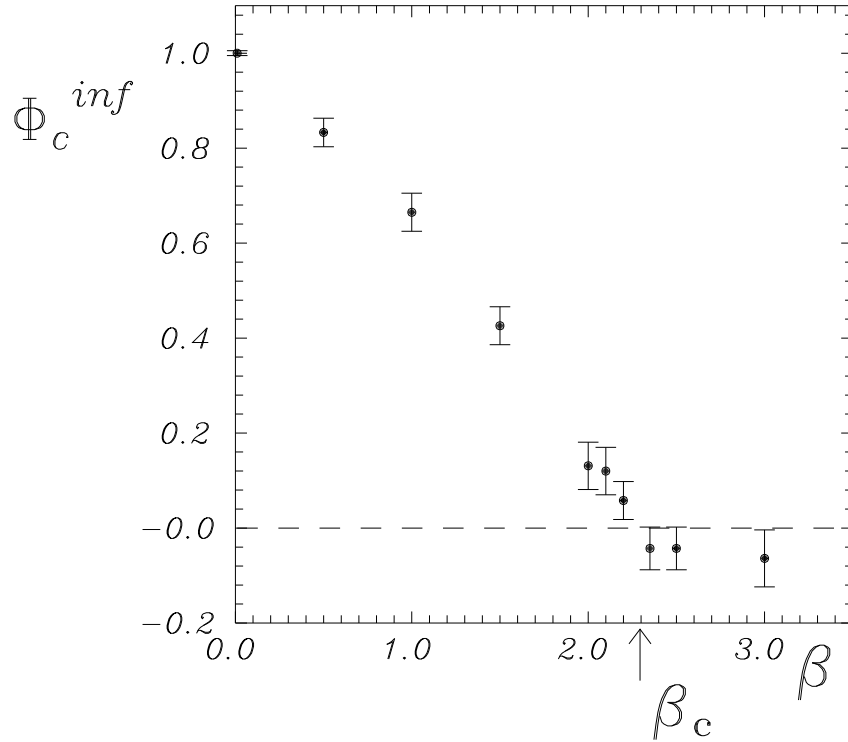
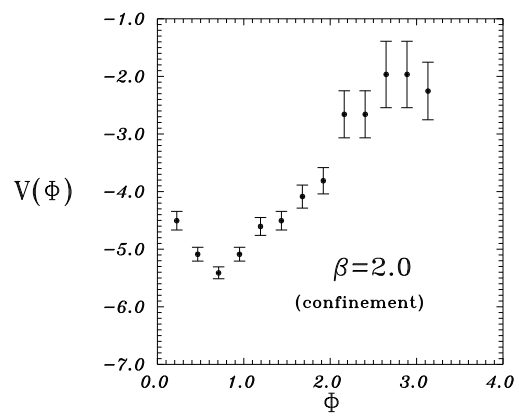
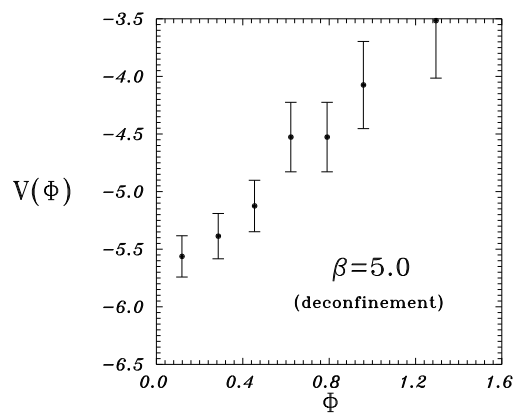


Figure 3: The dependence of Φ_c^{inf} on β .



(a)



(b)